MATH 2260 Calculus II
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Name: Answers
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Quiz No. 5

Show all of your work, label your answers clearly, and do not use a calculator.

Problem 1 What are the Taylor series for $\cos(x)$, $\sin(x)$, and e^x centered at $a = 0$? (You do not need to show work for this if you have them memorized.)

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

Problem 2 Compute the Taylor polynomial of degree 3 of $f(x) = 3x^2 + \cos(x) + e^{2x}$ centered at $a = 0$.

By computation:

$$f(x) = 3x^2 + \cos(x) + e^{2x}, \quad f(0) = 2$$
$$f'(x) = 6x - \sin(x) + 2e^{2x}, \quad f'(0) = 2$$
$$f''(x) = 6 - \cos(x) + 4e^{2x}, \quad f''(0) = 9$$
$$f'''(x) = \sin(x) + 8e^{2x}, \quad f'''(0) = 8$$

$$\Rightarrow \sum_{k=0}^{3} \frac{f^{(k)}(a)}{k!}(x-a)^k = 2 + 2(x) + \frac{9}{2!}(x)^2 + \frac{8}{3!}(x)^3$$
$$= 2 + 2x + \frac{9}{2}x^2 + \frac{4}{3}x^3$$

By using the Taylor series for $\cos(x)$ and e^x:

$$f(x) = 3x^2 + \cos(x) + e^{2x}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$
$$e^{(2x)} = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \dots$$

$$\Rightarrow f(x) = 3x^2 + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) + \left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots\right.$$

Now truncate by cutting off all terms of order > 3

$$\Rightarrow \sum_{k=0}^{3} \frac{f^{(k)}(a)}{k!}(x-a)^k = 2 + 2x + \left(3x^2 - \frac{x^2}{2} + 2x^2\right) + \frac{4}{3}x^3$$

Problem 3 Determine for what values of x the series converges

$$\sum_{n=1}^{\infty} \left(\frac{-5x}{n} \right)^n$$

Look at where the series converges absolutely: $\sum_{n=1}^{\infty} \left| \left(-\frac{5x}{n} \right)^n \right|$

Root test: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left| \left(-\frac{5x}{n} \right)^n \right|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{5x}{n} \right)^n}$

$= \lim_{n\to\infty} \frac{5x}{n} = 0 < 1$ for all x, so

the series converges absolutely for all x.

Problem 4 Use series to find the limit

$$\lim_{x\to 0} \frac{3x + 5\sin(2x)}{x}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\Rightarrow \sin(2x) = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots$$

$$= 2x - \frac{8}{6}x^3 + \frac{32}{120}x^5 - \cdots$$

$$\Rightarrow \lim_{x\to 0} \frac{3x + 5\sin(2x)}{x} = \lim_{x\to 0} \frac{3x + 5\left(2x - \frac{4}{3}x^3 + \frac{32}{120}x^5 - \cdots\right)}{x}$$

$$= \lim_{x\to 0} \left(13 - \frac{20}{3}x^2 + \frac{32}{120}x^4 - \cdots\right)$$

$$= 13$$