MATH 2260 Calculus II
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**Name:** Answers
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# Practice Quiz No. 5

Show all of your work, label your answers clearly, and do not use a calculator.

**Problem 1** What are the Taylor series for $\cos(x)$, $\sin(x)$, and $e^x$ (You do not need to show work for this if you have them memorized).

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

**Problem 2** Compute the Taylor polynomial of degree 3 of $f(x) = 5x^2 + 6x + 2 + e^x$ centered at $a = 0$.

Because $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$,

$$f(x) = 5x^2 + 6x + 2 + \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)$$

To get $P_3(x)$, truncate at the $x^3$ term

$$\Rightarrow \quad P_3(x) = (2+1) + (x + 6x) + \left(5x^2 + \frac{x^2}{2}\right) + \frac{x^3}{3!}$$

$$= 3 + 7x + \frac{11}{2}x^2 + \frac{1}{6}x^3$$

**Problem 3** Compute the Taylor polynomial of degree 4 of $f(x) = (5x^2 + 6x + 2)(e^x)$ centered at $a = 0$.

Because $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$

$$f(x) = (5x^2+6x+2)\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\dots\right),$$

to get $P_4(x)$ truncate the terms past order 4

$$P_4(x) = (5x^2+6x+2)(1) + (5x^2+6x+2)(x) + (5x^2+6x+2)\left(\frac{x^2}{2}\right) + \left(\cancel{5x^2} + 6x + 2\right)\left(\frac{x^3}{6}\right) + \left(\cancel{5x^2}+\cancel{6x}+2\right)\left(\frac{x^4}{24}\right)$$

order 5, so eliminate

order is higher than 4, so eliminate

**Problem 4** Compute the Taylor polynomial of degree 2 of $f(x) = 5x^2 + 6x + 2$ centered at $a = 2$.

$$P_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(2)}{k!}(x-2)^k$$

$$f(2) = 5(4) + 6(2) + 2 = 34$$
$$f'(2) = 10(2) + 6 = 26$$
$$f''(2) = 10$$

$$\Rightarrow P_2(x) = 34 + \frac{26}{1!}(x-2) + \frac{10}{2!}(x-2)^2$$

$$= 34 + 26(x-2) + 5(x-2)^2$$

**Problem 5** Determine the values of $x$ for which the Taylor series of $f(x) = \frac{1}{1+2x}$ converges, using $a = 0$.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$f(x) = \frac{1}{1+2x}$, notice that this is the sum of the geometric series with $a=1$, $r=-2x$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} (-2x)^k \Rightarrow \text{converges absolutely when } |-2x|<1$$

converges when $|x| < \frac{1}{2}$

Check $x = \frac{1}{2}$ and $x = \frac{-1}{2}$ by hand, $\sum (-1)^k$, $\sum 1^k$ both diverge

**Problem 6** Determine the values of $x$ for which the series $\sum_{n=1}^{\infty} \left(\frac{x^2-3}{5}\right)^n$

Geometric series $\Rightarrow$ converges absolutely when $\left|\frac{x^2-3}{5}\right| < 1$

$$\Rightarrow |x^2-3| < 5$$

$$\Rightarrow -5 < x^2-3 < 5$$

$$\Rightarrow -2 < x^2 < 8$$

$$\Rightarrow x^2 < 8 \quad \text{because } x^2 \geq 0$$

$$\Rightarrow -\sqrt{8} < x < \sqrt{8}$$

Check endpoints manually: $x = \pm\sqrt{8} \Rightarrow \sum \left(\frac{8-3}{5}\right)^n = \sum 1^n$ diverges

**Problem 7** What's the best bound you can find on the error of the polynomial $1 + \frac{x}{2}$ compared with $\sqrt{1+x}$? on $[-1/2, 1/2]$?

On $[-1/2, -1/2]$, $|x| \leq \frac{1}{2}$, and $1+\frac{x}{2} = P_1(x)$ for $\sqrt{1+x}$

$\Rightarrow |R_n(x)| \leq \frac{M}{2!}|x|^2$, where $|f''(x)| \leq M$ on $[-1/2, 1/2]$

$f'(x) = \frac{1}{2}(1+x)^{-1/2} \rightarrow |f''(x)| = \left|\frac{-1}{4}(1+x)^{-3/2}\right| = (1/4)(1+x)^{-3/2} \Rightarrow \frac{\partial}{\partial x}(f''(x)) = \frac{-3}{8}(1+x)^{-5/2}$

$= \frac{-3}{8}\left(\frac{1}{(1+x)^{5/2}}\right)$

$< 0$ on $[-1/2, 1/2]$,

so maximum of $|f''(x)|$ on $[-1/2, 1/2]$ is at $x = \frac{-1}{2}$

$\Rightarrow \left|f''\left(\frac{-1}{2}\right)\right| = \frac{1}{4}\left(1-\frac{1}{2}\right)^{-3/2}$

$\approx .707$

$\Rightarrow |R_n(x)| \leq \frac{.707}{2}\left(\frac{1}{2}\right)^2$

$= \frac{.707}{2^3}$

$\approx .088$

**Problem 8** Find the Taylor series for $\cos(5x^2)$ centered around $a = 0$.

$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

$\Rightarrow \cos(5x^2) = 1 - \frac{(5x^2)^2}{2} + \frac{(5x^2)^4}{4!} - \frac{(5x^2)^6}{6!} + \ldots$

**Problem 9** Use series to find the limit:

$$\lim_{t\to 0} \frac{1 - 3\sin(t) - t^3}{t^3}$$

$$\lim_{t\to 0} \frac{1 - 3\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots\right) - t^3}{t^3}$$

$$= \lim_{t\to 0} \frac{1 - 3t + \left(\frac{t^3}{2} - t^3\right) + \frac{t^5}{5!} - \frac{t^7}{7!} - \ldots}{t^3}$$

$$= \lim_{t\to 0} \left(\frac{1}{t^3} - \frac{3}{t^2} + \left(-\frac{1}{2}\right) + \frac{t^2}{5!} - \frac{t^4}{7!} + \ldots\right) = \text{DNE}$$

**Problem 10** Use series to find the limit:

$$\lim_{t\to 0} \frac{2t^5 - 4\cos(t) + t^3}{3t^6}$$

$$= \lim_{t\to 0} \frac{2t^5 - 4\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \ldots\right) + t^3}{3t^6}$$

$$= \lim_{t\to 0} \left(\frac{2}{3t} + \frac{1}{3t^3} - \frac{4}{3t^6} + \frac{4}{2t^4} - \ldots\right) = \text{DNE}$$