MATH 2260 Calculus II
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Practice Final Exam (Second Half Only)

Show all of your work, label your answers clearly, and do not use a calculator.

NOTE: The final exam is cumulative, but this practice exam is not. This practice exam only covers material after the midterm. To study for the portion of the final exam from before the midterm, study the practice midterm and the midterm.

Problem 1 Evaluate the following limits. Note that some of these should be coming directly from the table of common limits back in Section 9.1, some you may have to use series for. You do not have to show work for this problem:

a $\lim_{n\to\infty} \frac{\ln(n)}{n} =$ 0

b $\lim_{n\to\infty} n^{1/n} =$ 1

c $\lim_{n\to\infty} \frac{500^n}{n!} =$ 0

d $\lim_{n\to\infty} \frac{500^n}{n!} =$ 0

e $\lim_{x\to 0} \frac{3x^2\cos(x)}{x^2} =$ 3

f $\lim_{x\to 0} \frac{5x\sin(x)}{x^2} =$ 5

Problem 2 Determine whether or not the following series converge or diverge (You do not have to show work for this problem):

a $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ D

b $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ D

c $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ D

d $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$ C

e $\displaystyle\sum_{n=1}^{\infty} 12\left(\frac{3}{4}\right)^n$ C

f $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ C

g $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2+4n+2)}$ D

Problem 3 Find the function of x that the given series converges to, and state where the series converges absolutely, converges conditionally, and diverges:

$$\sum_{n=1} 3\left(\frac{x^2-3}{6}\right)^n$$

Geometric series with $a = 3$, $r = \frac{x^2-3}{6}$

Starts at $n=1$, so $\sum_{n=1}^{\infty} 3\left(\frac{x^2-3}{6}\right)^n = \frac{ar}{1-r} = \frac{3\left(\frac{x^2-3}{6}\right)}{1-\left(\frac{x^2-3}{6}\right)}$

Converges absolutely when $|r|<1$ (Geometric series test)

$\Rightarrow \left|\frac{x^2-3}{6}\right| < 1 \Rightarrow -1 < \frac{x^2-3}{6} < 1$

$\Rightarrow -6 < x^2 - 3 < 6 \rightarrow -3 < x^2 < 9$

$\Rightarrow x^2 < 9 \Rightarrow -3 < x < 3$

When $x = 3$ $\quad \sum 3\left(\frac{9-3}{6}\right)^n = \sum 3(1)^n$

Problem 4 Determine whether or not the given series converges or diverges and prove your answer, i.e. show all work:

$$\sum_{n=1} \frac{1}{n^{1/2} + n^2}$$

By comparison, $\frac{1}{n^{1/2}+n^2} \leq \frac{1}{n^2}$

and $\sum \frac{1}{n^2}$ converges, so $\sum \frac{1}{n^{1/2}+n^2}$ converges

as well.

Problem 5 Determine whether or not the following series converge, showing all work:

a $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$

Ratio test: $\lim_{n\to\infty} \left(\frac{2^{n+1}+5}{3^{n+1}}\right)\left(\frac{3^n}{2^n+5}\right) =$

$= \frac{1}{3}\lim_{n\to\infty} \left(\frac{2^{n+1}+5}{2^n+5}\right)\left(\frac{1/2^n}{1/2^n}\right) = \frac{1}{3}\lim_{n\to\infty}\left(\frac{2+\frac{5}{2^n}}{1+\frac{5}{2^n}}\right) = \frac{2}{3} < 1$ converges

b $\sum_{n=0}^{\infty} \left(\frac{1}{1+n}\right)^n$

Root test:

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \lim_{n\to\infty} \frac{1}{1+n} = 0 < 1 \Rightarrow$ converges

c $\sum_{n=0}^{\infty} \frac{3^n(2n)!}{(n!)^2}$ Ratio test: $\lim_{n\to\infty} \left(\frac{3^{n+1}(2n+2)!}{[(n+1)!]^2}\right)\left(\frac{(n!)^2}{3^n(2n)!}\right) =$

$= 3\lim_{n\to\infty} \frac{(2n+2)(2n+1)(2n)!\,(n!)^2}{(n+1)^2(n!)^2(2n)!} = 3\lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)^2} = 3\lim_{n\to\infty}\frac{4n^2+\dots}{n^2+\dots}$

$= 12 > 1$ diverges

should be $n=2$ →

d $\sum_{n=0}^{\infty} (-1)^n \frac{1}{\ln(n)}$ Alternating series test:

① $\frac{1}{\ln(n)} > 0$ for $n \geq 2$, ③ $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$ $\Rightarrow$ Converge.

② $\frac{1}{\ln(n)}$ decreasing because $\ln(n)$ increasing.

d $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{n^2}{1+3n+n^2}$

$\lim_{n\to\infty} \frac{n^2}{1+3n+n^2} = 1 \rightarrow \lim_{n\to\infty} (-1)^{n+1}\left(\frac{n^2}{1+3n+n^2}\right) = DNE$

Problem 6

a Given a sequence $\{a_n\}$ where $\lim_{n\to\infty} a_n = 0$, is it necessarily true that $\sum_{n=1}^{\infty} a_n$ converges? Give an example if not.

No! Example is $\sum_{n=1}^{\infty} \frac{1}{n}$

b Given a convergent series $\sum_{n=0}^{\infty} a_n$, is it necessarily true that $\lim_{n\to\infty} a_n = 0$? Give an example if not.

Yes.

Problem 7

a Find the Taylor series of $f(x) = \tan(x)$ of order 3 centered at $a = 0$.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x)^k = 0 + x + 0x^2 + 2x^3$$

$$f(0) = \tan(0) = 0$$
$$f'(0) = \sec^2(0) = 1$$
$$f''(0) = 2\sec(x)(\sec(x)\tan(x))\Big|_{x=0} = 0$$

$$f'''(0) = 2\sec^2(x)(\sec^2(x)) + 2(2\sec^2(x)\tan(x))\tan(x)\Big|_{x=0} = 2$$

b Find the Taylor series of $f(x) = \tan(x) + 3x^2 + \cos(x)$ of order 2 centered at $a = 0$.

From part a, 2^{nd} order Taylor polynomial for $\tan(x)$ is x

$$\Rightarrow P_2(x) = x + 3x^2 + \left(1 - \frac{x^2}{2}\right) = 1 + x + \frac{5}{2}x^2$$

c Find the Taylor series of $f(x) = 3x^2 + 2x + 5$ of order 2 centered at $a = 3$.

$$f(x) = 3x^2 + 2x + 5 \Rightarrow f(3) = 27 + 6 + 5 = 38$$
$$f'(x) = 6x + 2 \Rightarrow f'(3) = 20$$
$$f''(x) = 6 \Rightarrow f''(3) = 6$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = 38 + 20(x-3) + \frac{6}{2}(x-3)^2$$

Problem 8 Given $\vec{u} = (-2, 3)$ and $\vec{v} = (-1, -5)$, find the following:

a Find the lengths of $\vec{u}$ and $\vec{v}$.

$$\|\vec{u}\| = \sqrt{(-2)^2 + (3)^2} = \sqrt{4+9} = \sqrt{15}$$

$$\|\vec{v}\| = \sqrt{(-1)^2 + (-5)^2} = \sqrt{1+25} = \sqrt{26}$$

b Find the dot product of $\vec{u}$ and $\vec{v}$.

$$\vec{u} \cdot \vec{v} = (-2)(-1) + (3)(-5) = 2 - 15 = -13$$

c Find the angle θ between $\vec{u}$ and $\vec{v}$ using the dot product method.

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{-13}{\sqrt{15}\sqrt{26}}$$

d Find the projection of $\vec{u}$ onto $\vec{v}$, i.e. find $\text{proj}_{\vec{v}}\vec{u}$.

$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\right)\vec{v} = \left(\frac{-13}{26}\right)(-1, -5)$$

Problem 9 Given $\vec{u} = (4, -2, 3)$ and $\vec{v} = (-1, 2, -5)$, find the following:

a Find the cross product of $\vec{u}$ and $\vec{v}$, i.e. find $\vec{u} \times \vec{v}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 4 & -2 & 3 \\ -1 & 2 & -5 \end{vmatrix} = (10-6)i + (-3+20)j + (8-2)k$$

$$= (4, 17, 6)$$

b Parametrize the line between the points $\vec{u}$ and $\vec{v}$.

$$\vec{u} - \vec{v} = (4+1, -2-2, 3+5) = (5, -4, 8)$$

$$\Rightarrow f(t) = (5, -4, 8)t + (4, -2, 3)$$

c Parametrize the plane that goes through the point $\vec{v}$ and has normal vector $\dfrac{\vec{u}}{\|\vec{u}\|}$

$$((x, y, z) - \vec{\alpha}) \cdot \vec{n} = 0$$

$$\vec{\alpha} = \vec{v} = (-1, 2, -5)$$

$$\vec{n} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{(4, -2, 3)}{\sqrt{16+4+9}} = \frac{1}{\sqrt{29}}(4, -2, 3)$$

$$\Rightarrow (x+1, \cancel{y}-2, z+5) \cdot \left(\frac{1}{\sqrt{29}}\right)(4, -2, 3) = 0 \Rightarrow$$

$$\rightarrow 4(x+1) - 2(y-2) + 3(z+5) = 0$$