MATH 2250 Calculus I
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Name: Answers
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Test No. 1

Show all of your work, label your answers clearly, and do not use a calculator.

Problem 1 (10 points) State the (informal) definition of a limit. (Careful, it's surprising easy to make a logical mistake when rephrasing this definition.)

$\lim_{x \to a} f(x) = L$ if as x approaches a, $f(x)$ approaches L.

Problem 2 (10 points) Evaluate the limits below. You do not have to show any work for this problem.

a $\lim_{x \to 0} e^x = 1$

b $\lim_{x \to -\infty} e^x = 0$

c $\lim_{x \to \infty} \frac{4x^3 - 1}{7x^3 - 8x} = \frac{4}{7}$

d $\lim_{x \to -2} \frac{x+2}{x^2 - x - 6} = \lim_{x \to -2} \frac{x+2}{(x-3)(x+2)} = \lim_{x \to -2} \frac{1}{x-3} = \frac{-1}{5}$

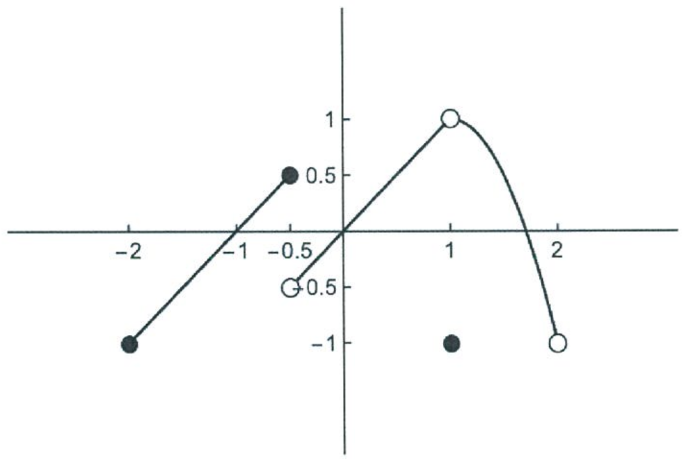
e $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

f $\lim_{x \to \infty} e^{-x} \cos(x) = 0$

Problem 3 (10 points) State the definition of what it means for a function $f(x)$ to be continuous at a point a (assume that a is not an endpoint of the domain).

$f(x)$ is continuous at a if $\lim_{x \to a} f(x) = f(a)$.

Problem 4 (15 points) Use the below graph to answer this question:



a Find $\lim_{x \to -1} f(x)$. $= 0$

b Find $\lim_{x \to -0.5^+} f(x)$. $= -0.5$

c Find $\lim_{x \to -0.5^-} f(x)$. $= 0.5$

d Find $\lim_{x \to 1} f(x)$. $= 1$

e Is $f(x)$ continuous at $x = -0.5$? Why or why not?

No, $\lim_{x \to -.5^+} f(x) \neq \lim_{x \to -.5^-} f(x)$

f Is $f(x)$ continuous at $x = -2$? Why or why not?

Yes, because for endpoints of the domain, we only need $\lim_{x \to -2^+} f(x) = f(-2)$

g Is $f(x)$ continuous? Why or why not?

No, because it is not continuous at every point in its domain.

Problem 5 (10 points) Justifying your work, evaluate $\lim_{x\to 1}\left(\frac{x-1}{x^2+3x-4}+5^x\right)$

$$= \lim_{x\to 1}\left(\frac{x-1}{x^2+3x-4}\right) + \lim_{x\to 1}\left(5^x\right)$$

$$= \lim_{x\to 1}\left(\frac{x-1}{(x-1)(x+4)}\right) + 5^1 \quad \text{, because } 5^x \text{ is a continuous function}$$

$$= \lim_{x\to 1}\left(\frac{1}{x+4}\right) + 5$$

$$= \frac{1}{5} + 5 \quad \text{, because } \frac{1}{x+4} \text{ is continuous}$$

Problem 6 (10 points) Justifying your work, find all vertical, horizontal, and slant asymptotes of $f(x) = \frac{3x^2+4x+1}{x^2-1}$

$$f(x) = \frac{(3x+1)(x+1)}{(x-1)(x+1)} = \begin{cases} \frac{3x+1}{x-1}, & x \neq \pm 1 \\ \text{undefined}, & x = \pm 1 \end{cases}$$

$\Rightarrow$ Horizontal asymptote at $y=3$ because $\lim_{x\to\pm\infty} f(x) = 3$

Vertical asymptote at $x=1$ because the denominator equals 0 at $x=1$ and the numerator is not zero.

No slant asymptotes, because $\lim_{x\to\infty} f(x) = 3$ and $\lim_{x\to -\infty} f(x) = 3$.

Problem 7

a (10 points) Find the slope of the tangent line (i.e. the instantaneous rate of change) for the function $f(x) = \frac{4}{2x+3}$ at the point $x = 1$, i.e. find

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{4}{2(x+h)+3} - \frac{4}{2x+3}}{h}$$

$$= \lim_{h \to 0} \frac{4(2x+3) - 4(2(x+h)+3)}{h(2x+3)(2(x+h)+3)}$$

$$= 4 \lim_{h \to 0} \frac{2x+3-2x-2h-3}{h(2x+3)(2(x+h)+3)}$$

$$= 4 \lim_{h \to 0} \frac{-2h}{h(2x+3)(2(x+h)+3)}$$

$$= -8 \lim_{h \to 0} \frac{1}{(2x+3)(2(x+h)+3)}$$

$$= -8 \frac{1}{(2x+3)^2}$$ Now plug in $x=1 \Rightarrow f'(1) = \frac{-8}{25}$

b (5 points) Now find the equation of the tangent line for the function $f(x) = \frac{4}{2x+3}$ at the point $x = 1$

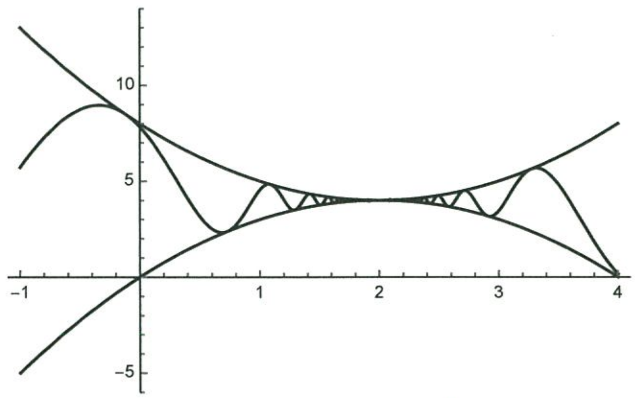
$$y - f(1) = f'(1)(x-1)$$

$$y - \frac{4}{5} = \left(\frac{-8}{25}\right)(x-1)$$

Problem 8 (10 points) The *squeeze theorem* states the following: If $f(x) \leq g(x) \leq h(x)$ for all real numbers x, and if $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = L$ as well.

Graphed below are the functions $(x-2)^2+4$, $-(x-2)^2+4$, and $((x-2)^2)\sin\left(\frac{10}{x-2}\right)+4$.

Use this information to find $\lim_{x\to 2}\left(((x-2)^2)\sin\left(\frac{10}{x-2}\right)+4\right)$ and justify your answer.



Because $g(x) = (x-2)^2\sin\left(\frac{10}{x-2}\right)+4$ is between $(x-2)^2+4$ and $-(x-2)^2+4$, using the squeeze theorem $\lim_{x\to 2} g(x) = 4$

Problem 9 (10 points) Prove, using the Intermediate Value Theorem, that the equation $10x^5+3x^3-2x+5=0$ has a solution on the interval $[-1,1]$.

$f(x) = 10x^5+3x^3-2x+5$ is continuous because it is a polynomial. Now check

$f(-1) = -10-3+2+5 = -6$

$f(1) = 10+3-2+5 = 16$

Because $y_0 = 0$ is between -6 and 16, by the IVT $10x^5+3x^3-2x+5=0$ for some c in $[-1,1]$.

Problem 10 (Bonus) (5 points) Given the function $f(x) = -2x^2 + 3$, find a function $g(a)$ that returns the x-intercept of the tangent line to the graph of $f(x)$ at the point $x = a$. (Hint: First find the derivative of $f(x)$ as a function of x. Then write out the expression for the tangent line at the point $x = a$. Then solve for the x-intercept, and express the x-intercept as a function of a).

Find the derivative $f'(x) = -4x$ however you want. Now, the tangent line at a point a is

$$y - f(a) = f'(a)(x-a)$$

The x-intercept is where $y = 0$, so

$$0 - f(a) = f'(a)(x-a)$$

$$\Rightarrow \frac{-f(a)}{f'(a)} = x - a$$

$$\Rightarrow a - \frac{f(a)}{f'(a)} = x$$

$$\Rightarrow a - \frac{-2a^2+3}{-4a} = x$$

$$\Rightarrow g(a) = a - \frac{2a^2-3}{4a}$$