MATH 2250 Calculus
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Name: Answers
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Practice Test No. 4

Show all of your work, label your answers clearly, and do not use a calculator.

Problem 1 (15 points) State both parts of the Fundamental Theorem of Calculus:

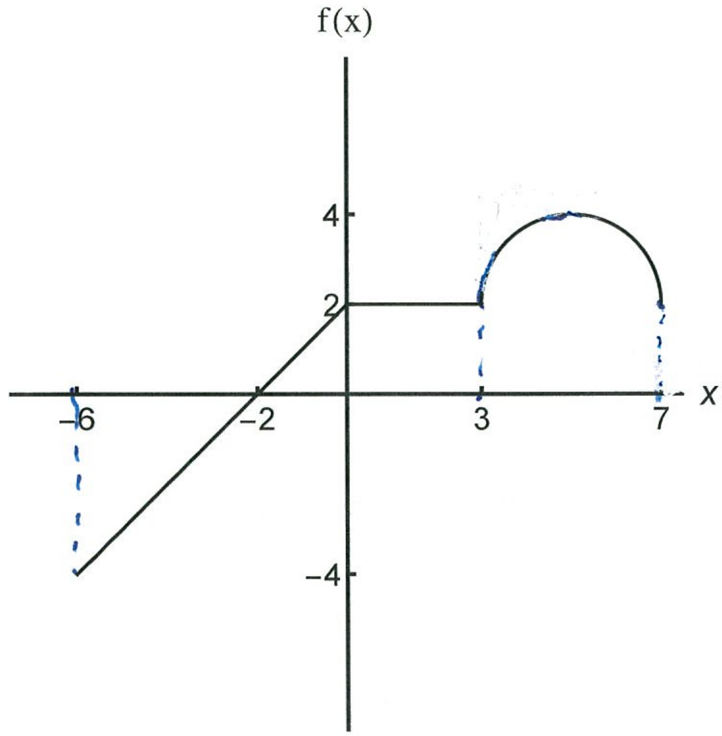
a Part I

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a, b), and $F'(x) = f(x)$.

b Part II

If f is continuous on $[a,b]$, and if F is an antiderivative of f on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Problem 2 (10 points) Find the definite integral $\int_{-6}^{7} f(x)dx$ using the graph of $f(x)$ given below. Show as much work as you can for partial credit. (The portion of the graph that looks like a semicircle is in fact a semicircle).



$$\int_{-6}^{7} f(x)\,dx = -\frac{1}{2}(4)(4) + \frac{1}{2}(2)(2) + (3)(2) + (4)(2) + \frac{1}{2}(\pi \cdot 2^2)$$

$$= -8 + 2 + 6 + 8 + 2\pi$$

$$= 8 + 2\pi$$

Problem 3 (25 points) Evaluate the following antiderivatives, definite integrals, and average values.

a $\int -2x^6 + \sqrt{x}\, dx$

$$= -2\left(\frac{x^7}{7}\right) + \frac{x^{3/2}}{3/2} + C$$

b $\int \frac{1}{t} + \sin(t)\, dt$

$$= \log(|t|) + (-\cos(t)) + C$$

d $\int_0^1 \sec(t)\tan(t) + 1\, dt$

$$\int \sec(t)\tan(t) + 1\, dt = \sec(t) + t + C$$

$$\Rightarrow \int_0^1 \sec(t)\tan(t) + 1\, dt = \left[\sec(t) + t\right]_0^1 = (\sec(1) + 1) - (\sec(0) + 0)$$

$$= \sec(1)$$

e Find the average value of $f(x) = \frac{3}{x^2} + 1$ on the interval $[2, 4]$.

$$(\text{Avg. value}) = \frac{1}{4-2}\int_2^4 \frac{3}{x^2} + 1\, dx$$

$$= \frac{1}{2}\left[3\left(\frac{x^{-1}}{-1}\right) + x\right]_2^4$$

$$= \frac{1}{2}\left(\left(\frac{-3}{4} + 4\right) - \left(\frac{-3}{2} + 2\right)\right) = \frac{1}{2}\left(\frac{3}{4} + 2\right) = \frac{11}{8}$$

Problem 5 (15 points) A small ball is stuck on the end of a spring, and the ball is bouncing up and down on the spring. The vertical acceleration function of the ball is $a(t) = \sin(t)$, the velocity at $t = \pi$ is $v(\pi) = 1$, and the position at $t = \pi$ is $s(\pi) = 3$, answer the following questions:

a Find the velocity function $v(t)$ of the particle.

$$a(t) = \sin(t) = \frac{\partial v}{\partial t} \Rightarrow v(t) = \int a(t)\, dt$$

$$v(t) = \int \sin(t)\, dt = -\cos(t) + C$$

And $v(\pi) = 1 \Rightarrow v(\pi) = -\cos(\pi) + C = 1$

$$\Rightarrow -(-1) + C = 1$$

$$\Rightarrow C = 0 \quad \Rightarrow v(t) = -\cos(t$$

b Find the position function $s(t)$ of the particle.

$$s(t) = \int \frac{\partial v}{\partial t}\, dt = \int -\cos(t)\, dt = -\sin(t) + C$$

And $s(\pi) = 3 = -\sin(\pi) + C$

$$\Rightarrow 3 = C$$

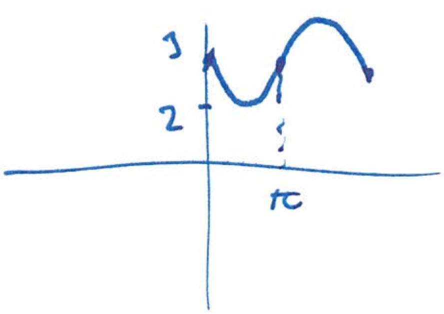
$$\Rightarrow s(t) = -\sin(t) + 3$$

c ~~How far does the ball move between $t = 0$ and $t = \pi$?~~

$$s(\pi) - s(0) = -\sin(\pi) + 3 - (-\sin(0) + 3)$$

$$= 0$$, So no change in position.

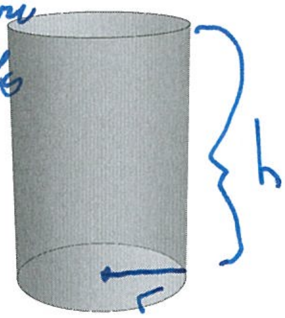
d.



So the distance travelled between $t=0$ and $t=\pi$ is 2.

Problem 6 (15 points) A container in the shape of a right circular cylinder without a top has a surface area 5π ft^2. What height and radius will maximize the volume? ~~(Hint: Remember to check to see if your critical point(s) are in the domain!)~~

Step 1: Picture with variables labled.



Step 2: Objective function

$$V(r,h) = \pi r^2 h$$

Step 3: Constraint equation

$$A = 5\pi = \pi r^2 + 2\pi r h$$

$$\Rightarrow h = \frac{5\pi - \pi r^2}{2\pi r} = \frac{5-r^2}{2r}$$

$$\Rightarrow V(r) = \pi r^2\left(\frac{5-r^2}{2r}\right) = \frac{\pi}{2}(5r - r^3)$$

Step 4: Domain is $(0, \sqrt{5}]$ because $r \geq 0$, and

$$h = \frac{5-r^2}{2r} \geq 0 \Rightarrow 5 - r^2 \geq 0 \Rightarrow 5 \geq r^2 \Rightarrow \sqrt{5} \geq r.$$

However, if $r = 0$, then $A = 5\pi = \pi(0^2) + 2\pi(0)h = 0$

So $r > 0$, not just $r \geq 0$.

Step 5: Maximize $V(r) = \frac{\pi}{2}(5r - r^3)$ on $(0, \sqrt{5}]$.

$V'(r) = \frac{\pi}{2}(5 - 3r^2)$ always exists $\Rightarrow$ set $0 = \frac{\pi}{2}(5 - 3r^2)$

$$\Rightarrow 5 = 3r^2 \Rightarrow \sqrt{\frac{5}{3}} = r.$$

Step 6:

0 $\quad \sqrt{\frac{5}{3}}$ $\quad \sqrt{5}$

$V'(r) > 0$ $\quad V'(r) < 0$

So $r = \sqrt{\frac{5}{3}}$ ft and $h = \left(\frac{5 - \frac{5}{3}}{2\sqrt{5/3}}\right)$ ft are the radius and height that will maximize the volume.

Problem 7 (15 points) For this problem, you will need the formula

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

a First use either high school geometry or the Fundamental Theorem of Calculus to compute the definite integral $\int_0^1 3x + 1dx$.

$$\int_0^1 3x+1dx = \left[3\left(\frac{x^2}{2}\right)+x\right]_0^1 = \left(\frac{3}{2}+1\right) - \left(\frac{3}{2}(0)+0\right) = \frac{5}{2}$$

b Using the definition and picking c_k as the right end point of the k^{th} interval, write an expression for the Riemann sum in terms of n, the number of rectangles the interval $[0, 1]$ is divided up into.

$$\sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n}\left(3\left(\frac{k}{n}\right)+1\right)\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{3}{n}k+1\right)$$

$$= \frac{1}{n}\left(\frac{3}{n}\left(\sum_{k=1}^{n}k\right)+\left(\sum_{k=1}^{n}1\right)\right) = \frac{3}{n^2}\left(\frac{n(n+1)}{2}\right)+\frac{1}{n}(n)$$

$$= \frac{3}{2}\left(\frac{n^2+n}{n^2}\right)+1 = \frac{3}{2}\left(1+\frac{1}{n}\right)+1$$

$$= \frac{5}{2}+\frac{3}{2}\left(\frac{1}{n}\right)$$

c Take the limit of the expression from part b to find the definite integral $\int_0^1 3x+1dx$ using the definition and picking c_k as the right end point of the k^{th} interval.

$$\lim_{n\to\infty}\left(\frac{5}{2}+\frac{3}{2}\left(\frac{1}{n}\right)\right) = \frac{5}{2}$$

Same as in part a.